The Role of the Area Averaging Approach to Derive Macroscopic Scaled Transport Equations for Hydrogel Capillaries

Kurt Dunham, Stephanie Jorgensen, Robby Sanders, and Pedro Arce
Department of Chemical Engineering

Introduction

- A hydrogel is a three-dimensional network of chemically or physically cross-linked polymers.
- Hydrogels have porous structures at the nanometer scale and are able to contain large amounts of water without changing their structure.
- Hydrogels are widely used for gel electrophoresis, which is a technique to separate biomolecules, such as proteins, nucleic acids, and pharmaceuticals, for industrial, biological, and environmental processes.
- During gel electrophoresis, biomolecules migrate through pores of varying shapes and sizes in the presence of an electric field and separate according to their size and charge.

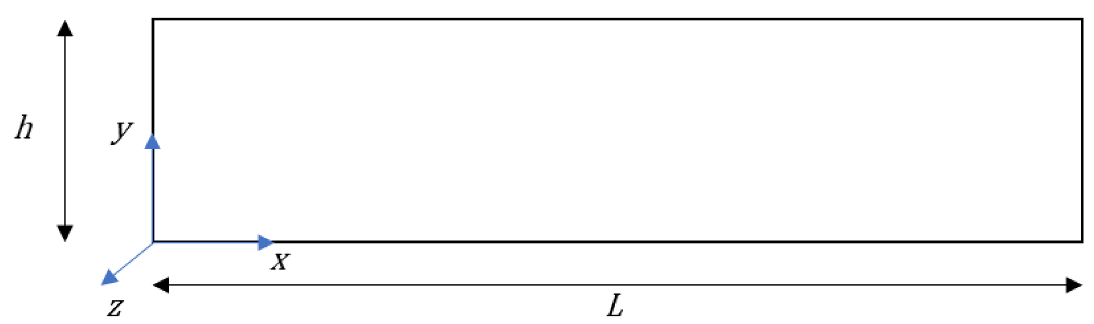

Figure 1: Model rectangular domain where gel electrophoresis occurs

Three governing equations determine the physics of the given system.

1. A form of the Hagen-Poiseuille velocity profile describes 1-D, incompressible flow in a rectangular channel with stationary boundaries.

$$v_x = \frac{\Delta P}{L}\frac{h^2}{2\mu}\left[\left(\frac{y}{h}\right)^2 - \frac{y}{h}\right] \qquad (1)$$

2. The Laplace equation is used to determine the electric potential in a 2-D system.

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0 \qquad (2)$$

3. The molar species continuity equation accounts for convective, diffusive, and electrostatic species transport and is used to determine the concentration profile.

$$\frac{\partial c}{\partial t} + v_x\left(\frac{\partial c}{\partial x} + \frac{\partial c}{\partial y}\right) = u\left(\frac{\partial}{\partial x} c\frac{\partial \varphi}{\partial x} + \frac{\partial}{\partial y} c\frac{\partial \varphi}{\partial y}\right) + D\left(\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2}\right) \qquad (3)$$

Where v_x is the velocity in the x-direction, μ is the Newtonian viscosity, ΔP is the pressure drop in the rectangular channel, L is the length of the channel, h is the height of the channel, φ is the electric potential, c is the molar concentration of the macromolecular species, u is the electrophoretic mobility and D is the diffusivity.

Methods: Overall Strategy

Through this project, techniques are being developed and systematized for the analysis of the microscopic transport equations that describe gel electrophoresis at a local level, within a capillary of the hydrogel material. Such equations are upscaled to macroscopic equations using a technique known as area averaging.

1. The area average of a general spatial function $f(x,y,z)$ can be defined as

$$\langle f(x,y,z)\rangle = \frac{\int_0^h \int_0^L f(x,y,z)\,dy\,dz}{\int_0^h \int_0^L dy\,dz} \qquad (4)$$

2. Develop a closure condition that uses a deviation variable that can be determined using geometrical or physical assumptions. The average value $\langle f\rangle$ is related to the deviation variable $\tilde{f}(x,y)$ and to its point values $f(x,y)$ by the following equation:

$$f(x,y) = \langle f\rangle + \tilde{f}(x,y) \qquad (5)$$

3. Test the validity of the assumptions by comparison to the exact solution if possible.

Results

1. Area Average of Electrical Potential Field

- The average electrical potential is found by applying equation (4) to the Laplace equation (2)

$$\frac{\partial^2\langle\varphi\rangle}{\partial x^2} + \frac{1}{h}\frac{\partial \varphi}{\partial y}\bigg|_0^h = 0 \qquad (6)$$

- Introduce the deviation variable using equation (5)

$$\varphi(x,y) = \langle\varphi\rangle + \tilde{\varphi}(x,y) \qquad (7)$$

- Substitute the expression for deviation (7) into the original Laplace equation (2) and subtract out the average expression (6).

$$\frac{\partial^2\tilde{\varphi}}{\partial x^2} + \frac{\partial^2\tilde{\varphi}}{\partial y^2} - \frac{1}{h}\frac{\partial \varphi}{\partial y}\bigg|_0^h = 0 \qquad (8)$$

- Use the long channel approximation where $L \gg h$ to neglect the deviation of the potential in the x-direction with respect to the potential in the y-direction.

$$\frac{\partial^2\tilde{\varphi}}{\partial y^2} - \frac{1}{h}\frac{\partial \varphi}{\partial y}\bigg|_0^h = 0 \qquad (9)$$

- Sauer, et al. apply the following boundary conditions (10) to equation (9) and solve for the average electric potential (11).

$$\varphi(x=0) = 0, \qquad \varphi(x=L) = K_1$$
$$\varphi(y=0) = \langle\varphi\rangle + \tilde{\varphi}(y=0) = 0 \qquad (10)$$
$$\varphi(y=h) = \langle\varphi\rangle + \tilde{\varphi}(y=h) = K_2$$

$$\langle\varphi\rangle = K_1\frac{\sinh\sqrt{12}\frac{x}{h}}{\sinh\sqrt{12}\frac{L}{h}} + \frac{K_2}{2}\left[\frac{\sinh\sqrt{12}\frac{L}{h} - \sinh\sqrt{12}\frac{x}{h} + \sinh\sqrt{12}\left(\frac{x}{h}-\frac{L}{h}\right)}{\sinh\sqrt{12}\frac{L}{h}}\right] \qquad (11)$$

2. Area Average of Concentration

- The average concentration is found by applying the area average equation (4) to the molar species continuity equation (3).

$$\frac{\partial\langle c\rangle}{\partial t} + \langle v\rangle\frac{\partial\langle c\rangle}{\partial x} + \frac{\partial\langle\tilde{c}\tilde{v}\rangle}{\partial x} = D\frac{\partial^2\langle c\rangle}{\partial x^2} + u\frac{\partial}{\partial x}\left(\langle c\rangle\frac{\partial\langle\varphi\rangle}{\partial x}\right) + \frac{u}{h}c\frac{\partial\varphi}{\partial y}\bigg|_0^h + u\frac{\partial}{\partial x}\left\langle\tilde{c}\frac{\partial\tilde{\varphi}}{\partial y}\right\rangle \qquad (12)$$

- Introduce the deviation variable using equation (5)

$$c(x,y) = \langle c\rangle + \tilde{c}(x,y) \qquad (13)$$

- Substitute the expression for deviation (13) into the molar species continuity equation (3) and subtract out the average expression (12).

$$\frac{\partial\tilde{c}}{\partial t} = D\left(\frac{\partial^2\tilde{c}}{\partial x^2} + \frac{\partial^2\tilde{c}}{\partial y^2}\right) - \tilde{v}\frac{\partial\langle c\rangle}{\partial x} - v\frac{\partial\tilde{c}}{\partial x} + \frac{\partial\langle\tilde{c}\tilde{v}\rangle}{\partial x} + u\left(\frac{\partial}{\partial x}\left[(\langle c\rangle + \tilde{c})\frac{\partial(\langle\varphi\rangle + \tilde{\varphi})}{\partial x}\right] + \frac{\partial}{\partial y}\left[(\langle c\rangle + \tilde{c})\frac{\partial\tilde{\varphi}}{\partial y}\right]\right) \qquad (14)$$

- Use the long channel approximation, the quasi-steady state approximation on the expression for the deviation field (14). Assume that the average concentration is much larger than the deviation of the concentration.

$$D\frac{\partial^2\tilde{c}}{\partial y^2} + u\frac{\partial}{\partial y}\left[(\langle c\rangle + \tilde{c})\frac{\partial\tilde{\varphi}}{\partial y}\right] = \tilde{v}\frac{\partial\langle c\rangle}{\partial x} \qquad (15)$$

- Sauer, et al. solve equation (15) by applying the following no-flux boundary conditions (16).

$$D\frac{\partial c}{\partial y}\bigg|_{y=0} + uc(0)\frac{\partial\varphi}{\partial y}\bigg|_{y=0} = 0$$
$$D\frac{\partial c}{\partial y}\bigg|_{y=h} + uc(h)\frac{\partial\varphi}{\partial y}\bigg|_{y=h} = 0 \qquad (16)$$

- Sauer, et al. use the following expressions (17) - (21) to determine the effective transport parameters (22) - (24).

$$\tilde{c}(x,y) = A(y)\langle c\rangle + B(y)\frac{\langle v\rangle}{D}\frac{\partial\langle c\rangle}{\partial x} \qquad (17)$$

$$A(y) = \frac{\exp\left(-\frac{u}{D}\tilde{\varphi}\right)}{\left\langle\exp\left(-\frac{u}{D}\tilde{\varphi}\right)\right\rangle} - 1 \qquad (18)$$

$$B(y) = \int_0^y G(y')\exp\left(-\frac{u}{D}[\tilde{\varphi}(y) - \tilde{\varphi}(y')]\right)dy' - \frac{\left\langle\int_0^y G(y')\exp\left(-\frac{u}{D}[\tilde{\varphi}(y) - \tilde{\varphi}(y')]\right)dy'\right\rangle}{\left\langle\exp\left(-\frac{u}{D}\tilde{\varphi}(y)\right)\right\rangle}\exp\left(-\frac{u}{D}\tilde{\varphi}(y)\right) \qquad (19)$$

$$G(y) = -3\left(\frac{2}{3}\frac{y^3}{h^2} - \frac{y^2}{h} + \frac{y}{3}\right) \qquad (20)$$

$$F(y) = -6\left(\left(\frac{y}{h}\right)^2 - \left(\frac{y}{h}\right) + \frac{1}{6}\right) \qquad (21)$$

$$D_{eff} = D - \frac{\langle v\rangle^2}{D}\langle F(y)B(y)\rangle \qquad (22)$$

$$V_{eff} = \langle v\rangle(1 + \langle F(y)A(y)\rangle) \qquad (23)$$

$$\frac{\partial\langle c\rangle}{\partial t} = D_{eff}\frac{\partial^2\langle c\rangle}{\partial x^2} - V_{eff}\frac{\partial\langle c\rangle}{\partial x} + u\frac{\partial}{\partial x}\left(\langle c\rangle\frac{\partial\langle\varphi\rangle}{\partial x}\right) \qquad (24)$$

Discussion

The migration of a biomolecule through a pore during gel electrophoresis may be modeled using these macro-transport equations, allowing for connection to macroscale parameters that are able to be measured experimentally. These parameters are directly related to fundamental transport and geometrical quantities of the system modeled.

References

1. Simhadri, J., H. A. Stretz, M. A. Oyanader and P.E. Arce, “Assessing Performance of Irregular Microvoids in Electrophoresis Separation,” I&E Chemistry Research, 54 (42), 10434-10441 (2015), (Contribution to Professor D. Ramkrisna’s Festschrift, Invited).
2. Sauer, S., B.R. Locke, and P.E. Arce, “Effect of Axial and Orthogonal Applied Electric Fields on Solute Transport in Poiseuille Flows: An Area Averaging Approach,” Ind. & Eng. Chem. Research, 34, 886 (1995).
3. Oyanader, M., P.E. Arce; "Role of Geometrical Dimensions in Electrophoresis Applications with Orthogonal Fields," Electrophoresis, 26, 2857 (2005).

Acknowledgments: TTU Department of Chemical Engineering